Student Number___________________

ABBOTSLEIGH

**AUGUST 2007**
**YEAR 12**
**ASSESSMENT 4**

**HIGHER SCHOOL CERTIFICATE**

**TRIAL EXAMINATION**

# Mathematics Extension 2

## General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

**Total marks – 120**

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

# Outcomes assessed

## HSC course

- **E1** appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
- **E2** chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
- **E3** uses the relationship between algebraic and geometric representations of complex numbers and of conic sections
- **E4** uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
- **E5** uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces and resisted motion
- **E6** combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
- **E7** uses the techniques of slicing and cylindrical shells to determine volumes
- **E8** applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
- **E9** communicates abstract ideas and relationships using appropriate notation and logical argument

Harder applications of the Extension 1 Mathematics course are included in this course. Thus the Outcomes from the Extension 1 Mathematics course are included.

### From the Extension 1 Mathematics Course

## Preliminary course

- **PE1** appreciates the role of mathematics in the solution of practical problems
- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives that require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

## HSC course

- **HE1** appreciates interrelationships between ideas drawn from different areas of mathematics
- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving projectiles, simple harmonic motion or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

**Total marks – 120**
**Attempt Questions 1-8**
**All questions are of equal value**

**Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.**

**Marks**

**QUESTION 1 (15 marks)**
**Use a SEPARATE writing booklet.**

(a) Find $\int \sin^3\theta \, d\theta$. **2**

(b) (i) Express $\dfrac{3x+1}{(x+1)(x^2+1)}$ in the form $\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+1}$. **2**

(ii) Hence find $\int \dfrac{3x+1}{(x+1)(x^2+1)}$. **2**

(c) Use the substitution $x = 2\sin\theta$, or otherwise, to evaluate $\int_1^{\sqrt{3}} \dfrac{x^2}{\sqrt{4-x^2}}\,dx$. **3**

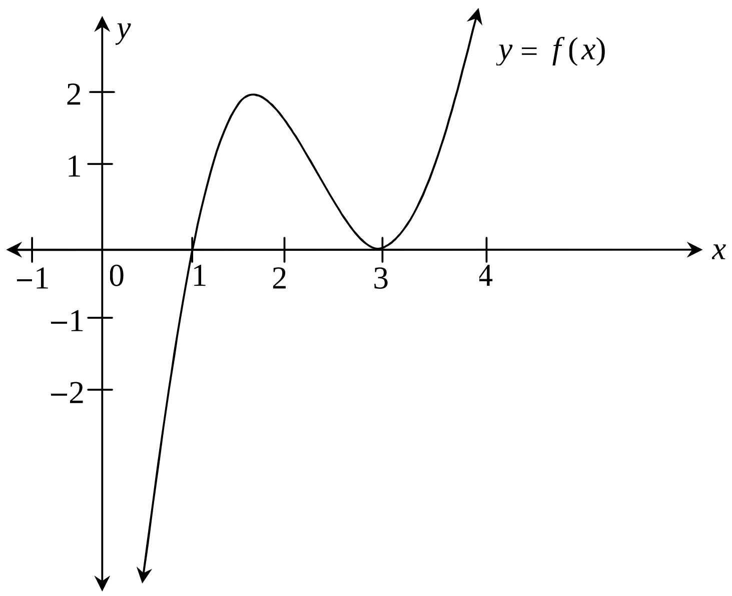
(d) Find $\int x^2\sqrt{3-x}\,dx$. **3**

(e) Evaluate $\int_0^1 \tan^{-1}\theta \; d\theta$. **3**

**Marks**

**QUESTION 2 (15 marks)**
**Start a new writing booklet.**

(a)




The diagram above is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = (f(x))^2$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \ln\left[f(x)\right]$ **2**

(iv) $y^2 = f(x)$ **2**

(b) (i) If $f'(x) = \dfrac{2-x}{x^2}$ and $f(1) = 0$, find $f''(x)$ and $f(x)$. **3**

(ii) Explain why the graph of $f(x)$ has only one turning point and find the value of the function at that point, stating whether it is a maximum or a minimum value. **2**

(iii) Show that $f(4)$ and $f(5)$ have opposite signs and draw a sketch of $f(x)$. **2**

**Marks**

**QUESTION 3 (15 marks)**
**Start a new writing booklet.**

(a) Express $(\sqrt{3}+i)^8$ in the form $x+iy$. **3**

(b) On an Argand diagram, sketch the region where the inequalities **3**

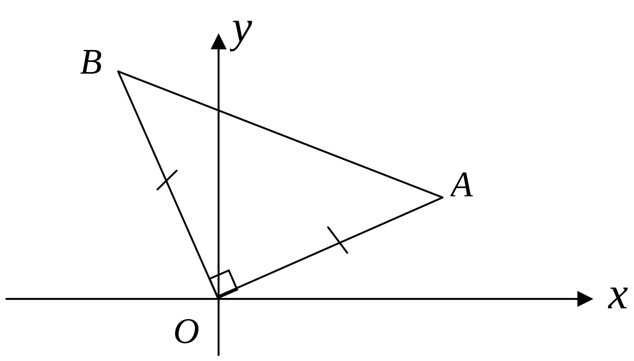
$$|z| \le 3 \text{ and } -\frac{2\pi}{3} \le \arg(z+2) \le \frac{\pi}{6} \text{ both hold.}$$

(c) Show that $\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$. **3**

(d) (i) Express $z = \dfrac{-1+i}{\sqrt{3}+i}$ in modulus-argument form. **2**

(ii) Hence evaluate $\cos\dfrac{7\pi}{12}$ in surd form. **2**

(e) The Argand diagram below shows the points $A$ and $B$ which represent the complex numbers $z_1$ and $z_2$ respectively.




Given that $\Delta BOA$ is a right-angled isosceles triangle, show that $(z_1+z_2)^2 = 2z_1z_2$. **2**

**Marks**

**QUESTION 4 (15 marks)**
**Start a new writing booklet.**

(a) If $z = 1 + i$ is a root of the equation $z^3 + pz^2 + qz + 6 = 0$ where $p$ and $q$ are real, find $p$ and $q$. **3**

(b) Show that if the polynomial $f(x) = x^3 + px + q$ has a multiple root, then $4p^3 + 27q^2 = 0$. **3**

(c) The base of a solid is the region in the first quadrant bounded by the curve $y = \sin x$, the $x$-axis and the line $x = \frac{\pi}{2}$. **3**

Find the volume of the solid if every cross-section perpendicular to the base and the $x$ – axis is a square.

(d) (i) Find the five roots of the equation $z^5 = 1$. Give the roots in modulus-argument form. **2**

(ii) Show that $z^5 - 1$ can be factorised in the form : **2**

$$z^5 - 1 = (z - 1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$$

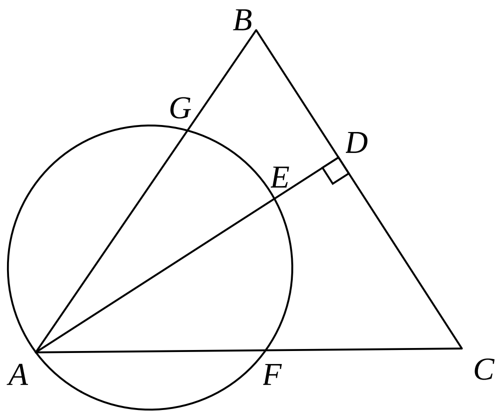
(iii) Hence show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **2**

**Marks**

**QUESTION 5 (15 marks)**
**Start a new writing booklet.**

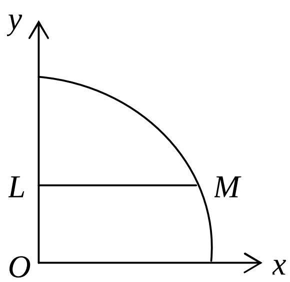
(a) The ellipse $(x-1)^2+\frac{y^2}{4}=1$ is rotated about the $y-$axis.
Use the method of slicing to find the volume of the solid formed by the rotation. **4**

(b) In the triangle $ABC$, $AD$ is the perpendicular from $A$ to $BC$. $E$ is any point on $AD$ and the circle drawn with $AE$ as diameter cuts $AC$ at $F$ and $AB$ at $G$. **4**




Prove $B, G, F$ and $C$ are concyclic.

(c) The diagram below shows the part of the circle $x^2+y^2=a^2$ in the first quadrant.




(i) If the horizontal line $LM$ through $L(0,b)$, where $0<b<a$, divides the area between the curve and the coordinates axes into two equal parts, show that

$$\sin^{-1}\frac{b}{a}+\frac{b\sqrt{a^2-b^2}}{a^2}=\frac{\pi}{4}.$$ **3**

(ii) If the radius of the circle is 1 unit, show that $b$ can be found by solving the equation
$\sin 2\theta=\frac{\pi}{2}-2\theta$, where $\theta=\sin^{-1}b$. **3**

(iii) Without attempting to solve the equation, how could $\theta$ (and hence $b$) be approximated? **1**

**Marks**

**QUESTION 6 (15 marks)**
**Start a new writing booklet.**

(a) An ellipse has equation $\frac{x^2}{4}+\frac{y^2}{3}=1$ with vertices $A(2,0)$ and $A'(-2,0)$. $P$ is a point $(x_1, y_1)$ on the ellipse.

(i) Find its eccentricity, coordinates of its foci, $S$ and $S'$, and the equations of its directrices. **3**

(ii) Prove that the sum of the distances $SP$ and $S'P$ is independent of the position of $P$. **2**

(iii) Show that the equation of the tangent to the ellipse at $P$ is $\frac{xx_1}{4}+\frac{yy_1}{3}=1$. **2**

(iv) The tangent at $P(x_1, y_1)$ meets the directrix at $T$. Prove that angle $PST$ is a right angle. **3**

(b) If $a+b+c=1$,

(i) Prove $a^2+b^2 \geq 2ab$. **1**

(ii) Prove $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$. **2**

(iii) Prove $(1-a)(1-b)(1-c) \geq 8abc$. **2**

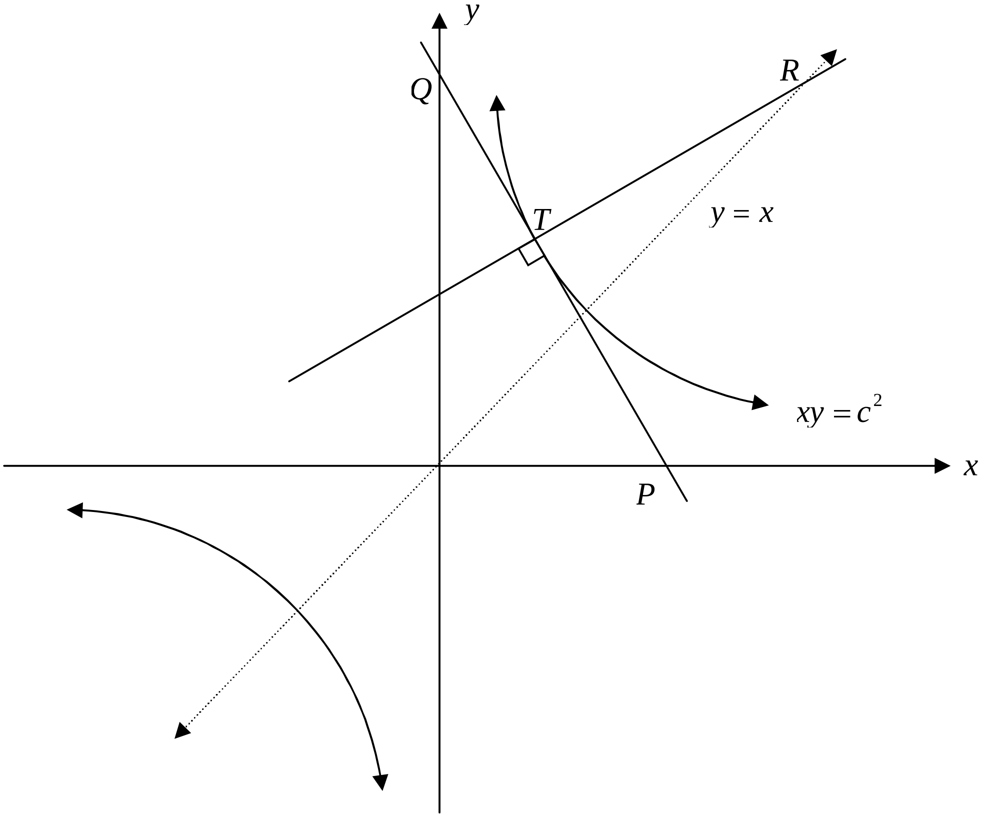
**Marks**

**QUESTION 7 (15 marks)**
**Start a new writing booklet.**

(a) The point $T(ct, \frac{c}{t})$ lies on the hyperbola $xy = c^2$.

The tangent at $T$ meets the $x$ – axis at $P$ and the $y$ – axis at $Q$.

The normal at $T$ meets the line $y = x$ at $R$.




*not to scale*

You may assume that the tangent at $T$ has equation $x + t^2 y = 2ct$.

(i) Find the coordinates of $P$ and $Q$. **2**

(ii) Find the equation of the normal at $T$. **2**

(iii) Show that the $x$ – coordinate of $R$ is $x = \frac{c}{t}(t^2 + 1)$. **2**

(iv) Prove that $\Delta PQR$ is isosceles. **3**

(b) (i) If $I_n = \int \frac{dx}{\left(x^2+1\right)^n}$ prove that $I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right]$. **4**

(ii) Hence evaluate $\int_0^1 \frac{dx}{\left(x^2+1\right)^2}$. **2**

**QUESTION 8 (15 marks)** **Marks**

**Start a new writing booklet.**

(a) A plane of mass $M$ kg on landing, experiences a variable resistive force due to air resistance of magnitude $Bv^2$ newtons, where $v$ is the speed of the plane. That is, $M\ddot{x} = -Bv^2$.

(i) Show that the distance $(D_1)$ travelled in slowing the plane from speed $V$ to speed $U$ under the effect of air resistance only, is given by: **4**

$$D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$$

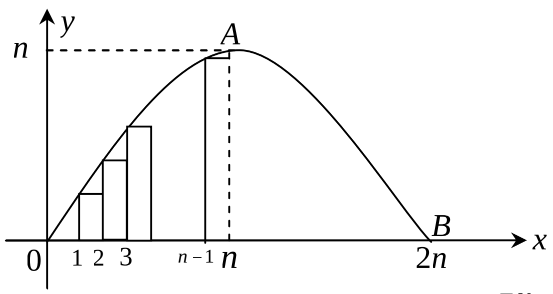
After the brakes are applied, the plane experiences a constant resistive force of $A$ Newtons (due to brakes) as well as a variable resistive force, $Bv^2$. That is, $M\ddot{x} = -(A + Bv^2)$.

(ii) After the brakes are applied when the plane is travelling at speed $U$, show that the distance $D_2$ required to come to rest is given by: **4**

$$D_2 = \frac{M}{2B}\ln\left[1 + \frac{B}{A}U^2\right].$$

(iii) Use the above information to estimate the total stopping distance after landing, for a 100 tonne plane if it slows from 90 $\mathrm{m/s^2}$ to 60 $\mathrm{m/s^2}$ under a resistive force of 125$v^2$ Newtons and is finally brought to rest with the assistance of a constant braking force of magnitude 75 000 Newtons. **2**

(b)




The diagram above represents the curve $y = n\sin\frac{\pi x}{2n},\ 0 \le x \le 2n,$ where $n$ is any integer $n \ge 2$.

The points $O(0,0)$, $A(n,n)$ and $B(2n,0)$ lie on this curve.

(i) By considering the areas of the lower rectangles of width 1 from $x = 0$ to $x = n$, prove that

$$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + .... + \sin\frac{\pi(n-1)}{2n} < \frac{2n}{\pi}.$$ **3**

(ii) Hence or otherwise, explain why $2n\sum_{r=1}^{n-1}\sin\frac{\pi r}{2n} < \frac{\pi n^2}{2}$. **2**

**END OF PAPER**

## QUESTION 1

a) $\int \sin^3\theta \, d\theta$

$= \int \sin\theta(1-\cos^2\theta)\,d\theta$

$= \int (\sin\theta - \sin\theta\cos^2\theta)\,d\theta$

$= -\cos\theta + \dfrac{\cos^3\theta}{3} + C$

b) (i) $a(x^2+1) + (bx+c)(x+1) = 3x+1$

Let $x=-1$, $2a+0=-2$

$\therefore a=-1$

coefficients of $x^2$: $a+b=0$

$\therefore b=1$

Let $x=0$ $\therefore -1\times 1 + c\times 1 = 1$

$\therefore c=2$

$\therefore \dfrac{3x+1}{(x+1)(x^2+1)} \equiv \dfrac{-1}{x+1} + \dfrac{x+2}{x^2+1}$

(ii) $\int \dfrac{3x+1}{(x+1)(x^2+1)}\,dx = \int\left(\dfrac{-1}{x+1} + \dfrac{x+2}{x^2+1}\right)dx$

$= -\ln(x+1) + \int\left(\dfrac{x}{x^2+1} + \dfrac{2}{x^2+1}\right)dx$

$= -\ln(x+1) + \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}x + C$

c) $I = \int_1^{\sqrt{3}} \dfrac{x^2}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta$ $\quad \sqrt{4-x^2} = \sqrt{4-4\sin^2\theta}$

$dx = 2\cos\theta\,d\theta$ $\quad = 2\cos\theta$

$x=\sqrt{3}$ $\quad \sin\theta = \frac{\sqrt{3}}{2}$, $\theta = \frac{\pi}{3}$

$x=1$ $\quad \sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

$\therefore I = \int_{\pi/6}^{\pi/3} \dfrac{4\sin^2\theta \times 2\cos\theta\,d\theta}{2\cos\theta}$

$= \int_{\pi/6}^{\pi/3} 4\sin^2\theta\,d\theta$

$= 2\int_{\pi/6}^{\pi/3}(1-\cos 2\theta)\,d\theta$

$= \left[2\theta - \sin 2\theta\right]_{\pi/6}^{\pi/3}$

$= \frac{2\pi}{3} - \sin\frac{2\pi}{3} - \frac{\pi}{3} + \sin\frac{\pi}{3}$

$= \frac{\pi}{3}$

d) $I = \int x^2\sqrt{3-x}\,dx$

$u = 3-x$

$du = -dx$

$\therefore I = \int (3-u)^2\sqrt{u} \times -du$

$= \int (9-6u+u^2)\times -\sqrt{u}\,du$

$= \int\left(-9\sqrt{u} + 6u^{3/2} - u^{5/2}\right)du$

$= -9u^{3/2}\times\frac{2}{3} + 6u^{5/2}\times\frac{2}{5} - u^{7/2}\times\frac{2}{7} + C$

$= -6(3-x)^{3/2} + \frac{12}{5}(3-x)^{5/2} - \frac{2}{7}(3-x)^{7/2} + C$

e) $I = \int_0^1 \tan^{-1}x\,dx$

let $u = \tan^{-1}x$ $\quad dV = dx$

$du = \dfrac{1}{1+x^2}\,dx$ $\quad V = x$

$\therefore I = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 x\times\dfrac{1}{1+x^2}\,dx$

$= \tan^{-1}1 - 0 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$

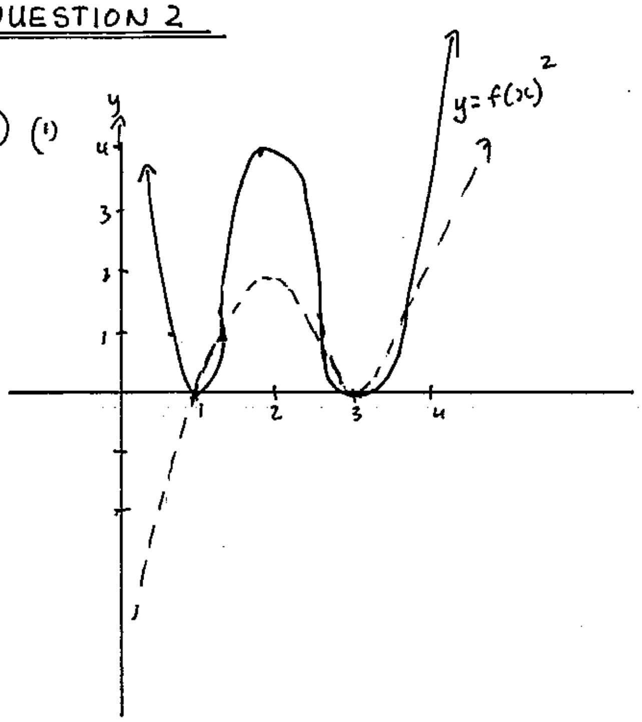
$= \frac{\pi}{4} - \frac{1}{2}\ln 2 + \frac{1}{2}\ln 1$

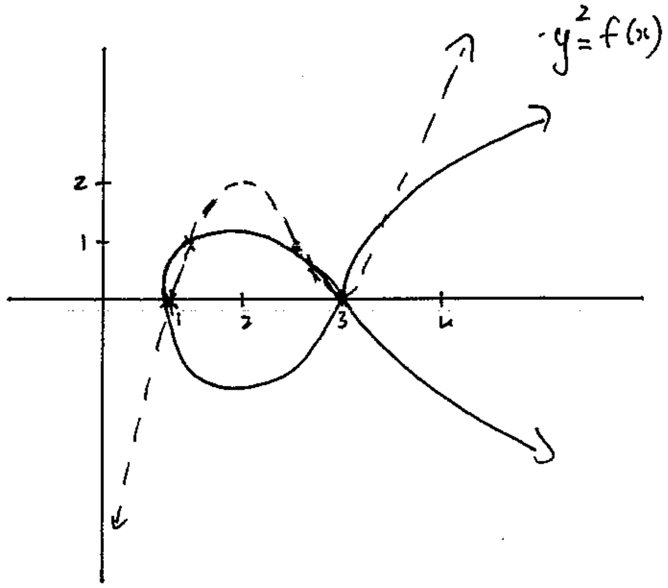
$= \dfrac{\pi - \ln 4}{4}$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

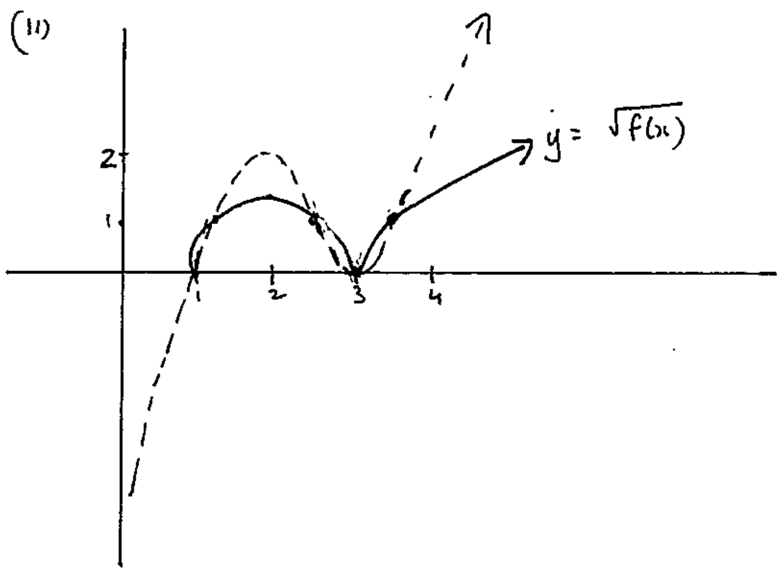
QUESTION 2

a) (i)

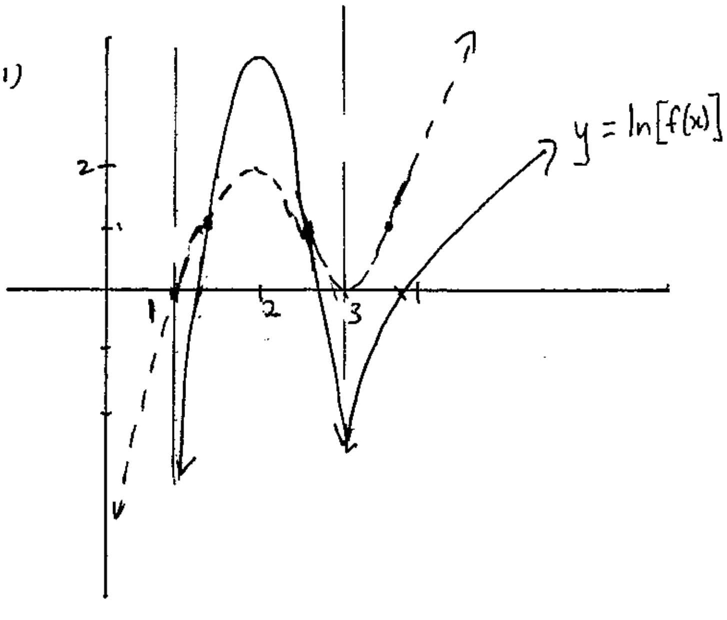



(ii)




(iii)




b) (i) $f'(x) = \frac{2-x}{x^2} = \frac{2}{x^2} - \frac{1}{x}$

$f(x) = \frac{2x^{-1}}{-1} - \ln x + C$

$f(1) = 0$

$0 = -2 - \ln 1 + C \Rightarrow C = 2$

$\therefore f(x) = -\frac{2}{x} - \ln x + 2$

$f''(x) = -4x^{-3} + x^{-2}$

$= -\frac{4}{x^3} + \frac{1}{x^2}$

(ii) stationary points at $f'(x) = 0$

$\therefore$ at $2 - x = 0 \Rightarrow x = 2$

at $x = 2$, $y = -1 - \ln 2 + 2$

$= 1 - \ln 2$

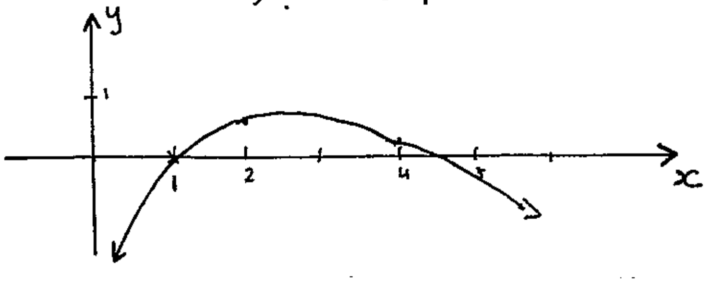
$f''(x) = -\frac{4}{8} + \frac{1}{4}$

$= -\frac{1}{4} < 0 \quad \therefore$ max tp at

$(2, 1 - \ln 2)$

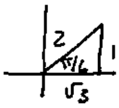
(iii) $f(4) \doteq 0.1137$

$f(5) \doteq -0.009$



QUESTION 3

a) $(\sqrt{3}+i)^8 = \left(2\,\text{cis}\,\frac{\pi}{6}\right)^8$




$= 2^8\ \text{cis}\,\frac{8\pi}{6}$

$= 256\ \text{cis}\left(-\frac{2\pi}{3}\right)$

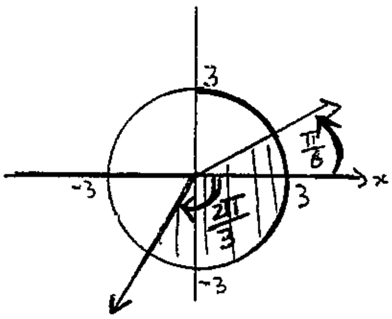
$= 256\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= 256\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

$= 256\left(-\frac{1}{2} - i\times\frac{\sqrt{3}}{2}\right)$

$= -128 - 128\sqrt{3}\,i$

b)




c) LHS $= \dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} \times \dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta + i\cos\theta}$

$= \dfrac{(1+\sin\theta)^2 + 2i\cos\theta(1+\sin\theta) + (i\cos\theta)^2}{(1+\sin\theta)^2 + (\cos\theta)^2}$

$= \dfrac{1 + 2\sin\theta + \sin^2\theta + 2i\cos\theta + 2i\cos\theta\sin\theta - \cos^2\theta}{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}$

$= \dfrac{2\sin^2\theta + 2\sin\theta + 2i\cos\theta\sin\theta + 2i\cos\theta}{2 + 2\sin\theta}$

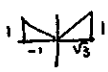
$= \dfrac{2\sin\theta(\sin\theta + 1) + 2i\cos\theta(\sin\theta+1)}{2(1+\sin\theta)}$

$= \dfrac{2(\sin\theta + i\cos\theta)(\sin\theta + 1)}{2(1+\sin\theta)}$

$= \sin\theta + i\cos\theta$

$=$ RHS

d) (i) $z = \dfrac{-1+i}{\sqrt{3}+i}$




$= \dfrac{\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)}{2\ \text{cis}\ \frac{\pi}{6}}$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{\sqrt{2}}\ \ \text{cis}\left(\frac{7\pi}{12}\right)$

$= \frac{1}{\sqrt{2}}\ \text{cis}\ \frac{7\pi}{12}$

d) (ii) $\dfrac{-1+i}{\sqrt{3}+i} \times \dfrac{\sqrt{3}-i}{\sqrt{3}-i} = \dfrac{-\sqrt{3} + i + i\sqrt{3} + 1}{3+1}$

$= \dfrac{1-\sqrt{3} + i(1+\sqrt{3})}{4}$

$\therefore\ \dfrac{1-\sqrt{3}}{4} + \dfrac{i(1+\sqrt{3})}{4} = \dfrac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

$\therefore\ \dfrac{1-\sqrt{3}}{4} = \frac{1}{\sqrt{2}}\cos\frac{7\pi}{12}$

$\cos\frac{7\pi}{12} = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

e) $z_2 = i z_1$

LHS $= (z_1 + z_2)^2$

$= z_1^2 + 2z_1 z_2 + z_2^2$

$= z_1^2 + 2z_1 \times i z_1 + (iz_1)^2$

$= z_1^2 + 2iz_1^2 - z_1^2$

$= 2iz_1^2$

$= 2z_1 \times iz_1$

$= 2z_1 z_2$

$=$ RHS

## QUESTION 4

a) If $z = 1+i$ is a root of $P(z)$

then $z = 1-i$ is also a root.

$\therefore z^3 + pz^2 + qz + 6 = (z-1-i)(z-1+i)(z-a)$

$= (z^2 - 2z + 1 + 1)(z-a)$

$= (z^2 - 2z + 2)(z-a)$

equating constant terms, $a = {}^{-}3$

$\therefore P(z) = (z^2 - 2z + 2)(z+3)$

$= z^3 + z^2 - 4z + 6$

$\therefore p = 1$ and $q = -4$

b) $f(x) = x^3 + px + q$

$f'(x) = 3x^2 + p$

If $f(x)$ has a multiple root then $f'(x) = 0$

$3x^2 + p = 0$

$x^2 = -\frac{p}{3}$

$\therefore f(x) = x(x^2 + p) + q = 0$

$x\left(-\frac{p}{3} + p\right) + q = 0$

$x \times \frac{2p}{3} + q = 0$

$x = \frac{-3q}{2p}$

$\therefore f(x) = \left(\frac{-3q}{2p}\right)^3 + p\left(\frac{-3q}{2p}\right) + q = 0$

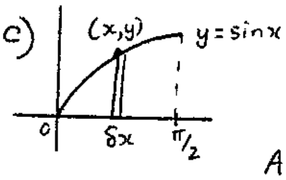
$\frac{-27q^3}{8p^3} - \frac{3q}{2} + q = 0$

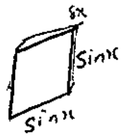
$-27q^3 - 12p^3q + 8p^3q = 0$

$27q^3 + 4p^3q = 0$

$4p^3 + 27q^2 = 0$

c)




Area of cross-section $= \sin^2 x$

$\therefore$ Volume of cross-section $= \sin^2 x\,\delta x$

Vol. of solid $= \lim_{\delta x \to 0} \sum_{x=0}^{\pi/2} \sin^2 x\,\delta x$

$= \int_0^{\pi/2} \sin^2 x\,dx$

$= \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\sin\pi}{2}\right) - \frac{1}{2}\left(0 - \frac{\sin 0}{2}\right)$

$= \frac{\pi}{4}$ cubic units

d) (i) Let $z = \cos\theta + i\sin\theta$

$\therefore z^5 = \cos 5\theta + i\sin 5\theta = 1$

$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

$\therefore$ Roots are $z_1 = \cos 0 + i\sin 0 \quad (=1)$

$z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$z_4 = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$

$z_5 = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$

(ii) $z^5 - 1 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)$

$= (z-1)\left(z^2 - z(z_2 + \bar{z}_2) + z_2\bar{z}_2\right)\left(z^2 - z(z_3 + \bar{z}_3) + z_3\bar{z}_3\right)$

$z_2 + \bar{z}_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

$= 2\cos\frac{2\pi}{5}$

$z_2\bar{z} = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)\left(\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}\right)$

$= \cos^2\frac{2\pi}{5} + \sin^2\frac{2\pi}{5}$

$= 1$

$\therefore z^5 - 1 = (z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$

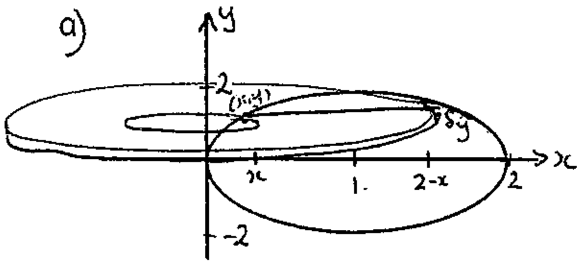
(iii) Sum of roots $= -\frac{b}{a}$

$\therefore 1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

# QUESTION 5

a)




Area of slice $= \pi R^2 - \pi r^2$

$= \pi(R-r)(R+r)$

$(x-1)^2 + \frac{y^2}{4} = 1$

$(x-1)^2 = \frac{4-y^2}{4}$

$x = 1 \pm \frac{\sqrt{4-y^2}}{2}$

$R - r = \sqrt{4-y^2}$ $\quad\quad R + r = 2$

Area of slice $= \pi(\sqrt{4-y^2}) \times 2$

$= 2\pi\sqrt{4-y^2}$

Vol of slice $= 2\pi\sqrt{4-y^2}\,\delta y$

Vol of solid $= \lim_{\delta y \to 0} \sum_{y=-2}^{2} 2\pi\sqrt{4-y^2}\,\delta y$

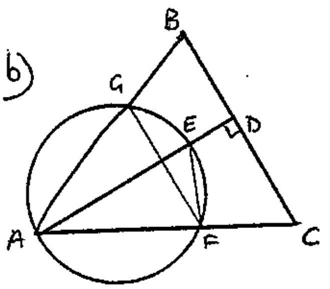
$= 2\pi \times 2 \int_0^2 \sqrt{4-y^2}\,dy$

Now $\int_0^2 \sqrt{4-y^2}\,dy = \frac{\pi \times 2^2}{4}$ since it represents the area of a quarter circle, radius 2 units.

$\therefore$ Vol of solid $= 2\pi \times 2 \times \pi$

$= 4\pi^2$ cubic units.

b)




$\angle AGF = \angle AEF$ ($\angle^s$ in same segment)

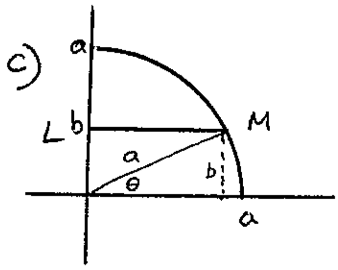
EDFC is a cyclic quad with diameter EC since $\angle EDC = 90°$

$\therefore \angle AEF = \angle DCF$ (ext. $\angle$ of a cyclic quad.)

$\therefore \angle AGF = \angle DCF$

$\therefore$ GBFC is a cyclic quad since its ext. $\angle$ is equal to the opposite interior $\angle$.

c)




$x = \sqrt{a^2 - y^2}$

$\sin\theta = \frac{b}{a}$

$\theta = \sin^{-1}\frac{b}{a}$

(i) Area of quadrant $= \frac{\pi a^2}{4}$

$\therefore$ Area of lower half $= \frac{\pi a^2}{8}$

$=$ sector + triangle

$\therefore \frac{1}{2}a^2\theta + \frac{1}{2}b\sqrt{a^2-b^2} = \frac{\pi a^2}{8}$

$a^2 \sin^{-1}\frac{b}{a} + b\sqrt{a^2-b^2} = \frac{\pi a^2}{4}$

$\sin^{-1}\frac{b}{a} + \frac{b}{a^2}\sqrt{a^2-b^2} = \frac{\pi}{4}$

(ii) If $a = 1$, then

$\sin^{-1} b + b\sqrt{1-b^2} = \frac{\pi}{4}$

$\theta + \sin\theta\sqrt{1-\sin^2\theta} = \frac{\pi}{4}$

$\theta + \sin\theta \times \cos\theta = \frac{\pi}{4}$

$\theta + \frac{1}{2}\sin 2\theta = \frac{\pi}{4}$

$\therefore \sin 2\theta = \frac{\pi}{2} - 2\theta$

(iii) Could use Newton's method to solve this equation.

(OR halving the interval method OR graph $y = \sin 2\theta$ & $y = \frac{\pi}{2} - 2\theta$ and find their points of intersection).

## QUESTION 6

a) $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$a = 2,\ b = \sqrt{3}$

(i) $b^2 = a^2(1 - e^2)$

$3 = 4(1 - e^2)$

$e = \frac{1}{2}$

Foci at $(\pm ae, 0) = (\pm 1, 0)$

Directrices $x = \pm \frac{a}{e}$

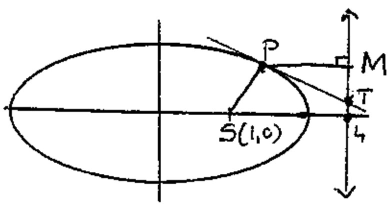
$= \pm \frac{2}{\frac{1}{2}}$

$\therefore x = \pm 4$

(ii) $P(x_1, y_1)$

By definition, $SP = e \times PM$

$\therefore$




$\therefore\ SP = e \times PM$

$S'P = e \times PM'$

$SP + S'P = e(PM + PM')$

$= \frac{1}{2}(MM')$

$= \frac{1}{2} \times 8$

$= 4$ which is a constant

$\therefore SP + SP'$ is independent of the position of P.

(iii) $\frac{2x}{4} + \frac{2y}{3}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-3x}{4y}$

At P, m of tangent $= \frac{-3x_1}{4y_1}$

$\therefore$ Eq'n of tangent is $y - y_1 = \frac{-3x_1}{4y_1}(x - x_1)$

$4yy_1 - 4y_1^2 = -3xx_1 + 3x_1^2$

$3xx_1 + 4yy_1 = 3x_1^2 + 4y_1^2$

$\frac{xx_1}{4} + \frac{yy_1}{3} = \frac{x_1^2}{4} + \frac{y_1^2}{3}$

$\therefore \frac{xx_1}{4} + \frac{yy_1}{3} = 1$

(iv) At T, $x = 4$ $\quad \therefore x_1 + \frac{yy_1}{3} = 1$

$\therefore T\left(4, \frac{3}{y_1}(1 - x_1)\right),\ S(1, 0)$

If $\angle PST$ is a right $\angle$, $m_{PS} \times m_{ST} = 1$

$LHS = \frac{y_1}{x_1 - 1} \times \frac{\frac{3}{y_1}(1 - x_1)}{4 - 1}$

$= \frac{y_1}{x_1 - 1} \times \frac{-3}{y_1}\frac{(x_1 - 1)}{3}$

$= -1$

$= RHS$

$\therefore \angle PST$ is a right angle

b) (i) $(a - b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$\therefore a^2 + b^2 \geq 2ab$

(ii) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$= \frac{ab + ac + bc}{abc}$

$= \frac{(ab + ac + bc)(a + b + c)}{abc}$ since $a + b + c = 1$

$= \frac{a^2b + a^2c + abc + ab^2 + abc + b^2c + abc + ac^2 + bc^2}{abc}$

$= \frac{3abc + c(a^2 + b^2) + a(b^2 + c^2) + b(a^2 + c^2)}{abc}$

$\geq \frac{3abc + c \times 2ab + a \times 2bc + b \times 2ac}{abc}$ from (i)

$= \frac{9abc}{abc}$

$= 9$ $\quad \therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$

(iii) $(1 - a)(1 - b)(1 - c)$

$= (b + c)(a + c)(a + b)$ since $a + b + c = 1$

$= (b + c)(a^2 + ab + ac + bc)$

$= a^2b + ab^2 + abc + b^2c + a^2c + abc + ac^2 + bc^2$

$= b(a^2 + c^2) + a(b^2 + c^2) + c(b^2 + a^2) + 2abc$

$\geq b \times 2ac + a \times 2bc + c \times 2ab + 2abc$

$= 8abc$

$\therefore (1 - a)(1 - b)(1 - c) \geq 8abc$

QUESTION 7

a) $xy = c^2$ $T(ct, \frac{c}{t})$

tangent at T: $x + t^2y = 2ct$

(i) $P(2ct, 0)$

$Q(0, \frac{2c}{t})$

(ii) at T, $m_{tan} = -\frac{1}{t^2}$ $\therefore m_{Norm} = t^2$

$\therefore$ eq'n of normal:

$y - \frac{c}{t} = t^2(x - ct)$

$y = t^2x - ct^3 + \frac{c}{t}$

(iii) at R, $y = x$

$\therefore x = t^2x - ct^3 + \frac{c}{t}$

$x(t^2 - 1) = c(t^3 - \frac{1}{t})$

$x(t^2 - 1) = \frac{c}{t}(t^4 - 1)$

$x = \frac{c}{t}\frac{(t^4 - 1)}{(t^2 - 1)}$

$\therefore x = \frac{c}{t}(t^2 + 1)$

(iv) $\therefore R\left(\frac{c}{t}(t^2+1), \frac{c}{t}(t^2+1)\right)$

If $\triangle$ OTR is isosceles,

OT = TR

LHS $= \sqrt{c^2t^2 + \frac{c^2}{t^2}}$

$= \sqrt{c^2(t^2 + \frac{1}{t^2})}$

RHS $= \sqrt{\left(ct - \frac{c}{t}(t^2+1)\right)^2 + \left(\frac{c}{t} - \frac{c}{t}(t^2+1)\right)^2}$

$= \sqrt{c^2(t - t + \frac{1}{t})^2 + c^2(\frac{1}{t} - t - \frac{1}{t})^2}$

$= \sqrt{c^2(\frac{1}{t^2} + t^2)}$

= LHS

$\therefore$ $\triangle$OTR is isosceles

b) $I_n = \int \frac{dx}{(x^2+1)^n}$ let $u = (x^2+1)^{-n}$

(i) $du = -n \times 2x(x^2+1)^{-n-1}\,dx$

$dv = dx$

$\therefore v = x$

$\therefore I_n = x \times (x^2+1)^{-n} - \int x \times -n \times 2x(x^2+1)^{-n-1}\,dx$

$= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2\,dx}{(x^2+1)^{n+1}}$

$= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2+1-1}{(x^2+1)^{n+1}}\,dx$

$= \frac{x}{(x^2+1)^n} + 2n\int \frac{dx}{(x^2+1)^n} - 2n\int \frac{dx}{(x^2+1)^{n+1}}$

$\therefore I_n = \frac{x}{(x^2+1)^n} + 2nI_n - 2nI_{n+1}$

$2nI_{n+1} = I_n(2n-1) + \frac{x}{(x^2+1)^n}$

putting $n+1 = n$, $\Rightarrow$ $n = n-1$

$2(n-1)I_n = I_{n-1}(2(n-1)-1) + \frac{x}{(x^2+1)^{n-1}}$

$I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right]$

(ii) $\int_0^1 \frac{dx}{(x^2+1)^2} = I_2$

$\therefore I_2 = \frac{1}{2}\left[\left(\frac{x}{x^2+1}\right)_0^1 + 1 \times I_1\right]$

$= \frac{1}{2}\left[\frac{1}{2} - 0 + \int_0^1 \frac{dx}{x^2+1}\right]$

$= \frac{1}{4} + \left[\frac{1}{2}\tan^{-1}x\right]_0^1$

$= \frac{1}{4} + \frac{1}{2} \times \left[\tan^{-1}1 - \tan^{-1}0\right]$

$= \frac{1}{4} + \frac{1}{2} \times \frac{\pi}{4}$

$= \frac{\pi + 2}{8}$

## QUESTION 8

a) (i) $M\ddot{x} = -Bv^2$

$$v\frac{dv}{dx} = \frac{-Bv^2}{M}$$

$$\frac{dv}{dx} = \frac{-Bv}{m}$$

$$\frac{dx}{dv} = -\frac{m}{Bv}$$

$$x = \int_V^U -\frac{m}{Bv}\,dv$$

$$\therefore D_1 = \left[-\frac{m}{B}\ln v\right]_V^U$$

$$= -\frac{M}{B}\ln U + \frac{m}{B}\ln V$$

$$D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$$

(ii) $M\ddot{x} = -(A+Bv^2)$

$$v\frac{dv}{dx} = \frac{-(A+Bv^2)}{M}$$

$$\frac{dv}{dx} = \frac{-(A+Bv^2)}{Mv}$$

$$\frac{dx}{dv} = \frac{-Mv}{A+Bv^2}$$

$$x = M\int_U^0 -\frac{v}{A+Bv^2}\,dv$$

$$\therefore D_2 = \left[-\frac{M}{2B}\ln(A+Bv^2)\right]_U^0$$

$$= \frac{-M}{2B}\ln A + \frac{M}{2B}\ln(A+BU^2)$$

$$= \frac{M}{2B}\ln\left(\frac{A+BU^2}{A}\right)$$

$$D_2 = \frac{M}{2B}\ln\left(1+\frac{B}{A}U^2\right)$$

(iii) $M = 100000$ kg, $V = 90$, $U = 60$

$Bv^2 = 125v^2 \quad \therefore B = 125, \quad A = 75000$

$\therefore$ total distance $= D_1 + D_2$

$$= \frac{100000}{125}\ln\left(\frac{90}{60}\right) + \frac{100000}{250}\ln\left(1+\frac{125}{75000}\times 60^2\right)$$

$$= 800\ln 1.5 + 400\ln(7)$$

$$= 1102.736\ldots$$

$$\doteqdot 1103 \text{ metres.}$$

b) (i) By adding area of rectangles,

$A_1$ = area under curve

$$\doteqdot n\sin\frac{\pi}{2n} + n\sin\frac{2\pi}{2n} + n\sin\frac{3\pi}{2n} + \ldots + n\sin\frac{(n-1)\pi}{2n}$$

By integrating to find the area under curve,

$$A_2 = \int_0^n n\sin\frac{\pi x}{2n}\,dx$$

$$= \left[n \times \frac{2n}{\pi} \times -\cos\frac{\pi x}{2n}\right]_0^n$$

$$= -\frac{2n^2}{\pi}\cos\frac{n\pi}{2n} + \frac{2n^2}{\pi}\cos 0$$

$$= \frac{-2n^2}{\pi}\cos\frac{\pi}{2} + \frac{2n^2}{\pi}\cos 0$$

$$= \frac{2n^2}{\pi}(0+1)$$

$$= \frac{2n^2}{\pi}$$

$A_1 < A_2$

$$\therefore n\left(\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \ldots + \sin\frac{\pi(n-1)}{2n}\right) < \frac{2n^2}{\pi}$$

$$\therefore \sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \ldots + \sin\frac{\pi(n-1)}{2n} < \frac{2n}{\pi}$$

(ii) From (i) $\displaystyle\sum_{r=1}^{n-1}\sin\frac{r\pi}{2n} < \frac{2n}{\pi}$

$$\therefore 2n\sum_{r=1}^{n-1}\sin\frac{r\pi}{2n} < \frac{4n^2}{\pi} = \frac{4\pi n^2}{\pi^2}$$

but $\frac{4}{\pi} < \frac{\pi}{2}$ (since $8 < \pi^2$)

$$\therefore \frac{4\pi n^2}{\pi \times \pi} < \frac{\pi}{2} \times \frac{\pi n^2}{\pi}$$

$$= \frac{\pi n^2}{2}$$

$$\therefore 2n\sum_{r=1}^{n-1}\sin\frac{r\pi}{2n} < \frac{\pi n^2}{2}$$